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2000 J. Phys. A: Math. Gen. 33 1017

(http://iopscience.iop.org/0305-4470/33/5/314)

J. Phys. A: Math. Gen. **33** (2000) 1017–1030. Printed in the UK PII: S0305-4470(00)06994-8

Combinatorics of lattice paths with and without spikes

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Received 17 August 1999

Abstract. We derive a series of results on random walks on a d-dimensional hypercubic lattice (lattice paths). We introduce the notions of terse and simple paths corresponding to the path having no backtracking parts (spikes). These paths label equivalence classes which allow a rearrangement of the sum over paths. The basic combinatorial quantities of this construction are given. These formulae are useful when performing strong-coupling (hopping parameter) expansions of lattice models. Some applications are described.

1. Introduction

The strong-coupling expansion is a useful analytical technique to study lattice models. In the context of lattice gauge theories it has been used since the early days of the subject to investigate the behaviour of the system far from the continuum limit [1]. This technique is related to the high-temperature expansions of classical statistical mechanics. In the case of matter fields (continuous spin variables) with nearest-neighbour interactions the technique involves a *hopping parameter expansion* giving rise to a representation of the free energy and the propagators (correlation functions) in terms of random walks (see [2] and references therein). In its application to gauge theories the contribution of each random walk includes the corresponding Wilson loop. Since the early calculations of this type [3] the special behaviour of *backtracking* paths was recognized. Backtracking occurs when the walk makes two consecutive opposite steps: the first in one direction and the next one in the reverse direction. Part of their special character is related to the fact that the expectation value of a Wilson loop is suppressed like an exponential of its area. At infinitely strong coupling and large N this leaves loops which are *pure backtrackers*. Actually, for unitary gauge fields the contribution of the backtracking part of a path is independent of the gauge fields themselves. The problem of summing over this type of path becomes independent of the expectation value of gauge fields and is a pure combinatorial problem. The problem was avoided, nevertheless, in some of the mentioned lattice QCD strong-coupling expansions [3, 4] by setting the Wilson parameter r equal to 1. This kills backtracking paths from quark propagators. Other strong-coupling calculations were performed at $r \neq 1$ by means of the effective potential method [5], in which the connection to the backtracking-path resummation problem is hidden. The problem was nevertheless addressed by several authors [6], and led to formulae with which one could reproduce the effective potential strong-coupling results. Recently, motivated by the strong-coupling expansion of supersymmetric Yang–Mills theory [7], we fell back into the problem. Unaware of the previous work on the subject†, we arrived independently at a derivation of the

† We thank J Smit, A Sokal and G Münster for pointing out to us some of the old papers on the subject.

0305-4470/00/051017+14$30.00

main formulae. In this paper we present the result of our investigation. Our paper contains a mathematically rigorous derivation of some of the previously known formulae, which are often disperse in the literature. Furthermore, using the same techniques, we derive some new combinatorial expressions which are useful in different contexts. Some new applications of the formulae are also presented. We believe that the applications of lattice path resummation are far from being exhausted. In addition, from the point of view of the combinatorial problem itself, there are some extensions which are useful in other situations [8] which are yet to be solved. We hope that our techniques can be a good starting point for attacking these problems.

This paper is written in a self-contained form. In the next section we introduce the basic notation and definitions. Since there is no standard precise terminology on the subject, we prefer to define and name all of the basic conceptual notions which are relevant for our work. Hence, we will refer to random walks as *lattice paths* and to backtracking parts as *spikes*. Section 2 contains the main results on resummation over pure backtracking parts. Section 3 gives the expression of a matrix generating function for paths without spikes (*terse paths*). This expression is useful for strong-coupling expansions of lattice gauge theories. In section 4, we consider some modifications which are useful when considering closed paths. In this case the notion of a *simple* path turns out to be useful. Formulae similar to those given in the previous two sections are given for the case of simple paths. Finally, in section 5 we give a few applications which exemplify the way in which the previous results enter in physical expressions within strong-coupling expansions. This includes the pure spike contribution to the free energy of a Gaussian model and the mesonic effective action. This is the only surviving part if the fields are coupled to a random $U(N)$ gauge field at large N [8]. The reader who is not interested in proofs can jump directly to the last section. Our main results are given in formulae (14), (18), (20), (32)–(34), (39)–(41) and (44).

2. Reducing paths

In this section we will introduce the basic notation and definitions. We will be working in arbitrary spacetime dimension d. Vector indices μ go from 0 to $d-1$. We will need to introduce an index set I with $2d$ elements. For every spacetime direction μ there are two elements μ and $\bar{\mu}$. They correspond to the two senses associated to each direction (forward and backward). Now consider our spacetime lattice $\mathcal{L} \equiv \mathbb{Z}^d$. We might associate to any element α in I a lattice vector $V(\alpha)$ as follows:

$$\mu \longrightarrow V(\mu) \equiv e^{(\mu)} \qquad \bar{\mu} \longrightarrow V(\bar{\mu}) \equiv -e^{(\mu)}$$

where $e^{(\mu)}$ is the unit vector in the μ-direction. Given one element $\alpha \in I$ the element $\bar{\alpha}$ denotes the oppositely oriented one ($\bar{\bar{\mu}} = \mu$).

Now we proceed to give a few definitions.

Definition 1. *A* lattice path *of length L is an element $\gamma \equiv (n, \vec{\alpha}) \in \mathcal{L} \times I^L$. The point $n \in \mathcal{L}$ is the* origin *of the path, and $\vec{\alpha}$ is the* path sequence*, specifying the steps to take to describe the path.*

The *endpoint* of a path $(n, \alpha_1, \dots, \alpha_L)$ is given by the lattice point $m = n + V(\vec{\alpha}) = n + \sum_{i=1}^{L} V(\alpha_i)$. We can now introduce the following nomenclature for the set of paths. Let $\mathcal{S}_L(n)$ be the space of all paths with origin n and length L. $\mathcal{S}(n)$ labels the set of all paths with origin n and any length. We might also fix the origin and endpoint and write $\mathcal{S}_L(n \to m)$.

The total number of paths of length L is easy to count: $N(L) = (2d)^L$. For length $L = 0$ we will consider that there is a unique path with origin in n, which we will call the *path of zero length*. To any path $\gamma \equiv (n, \vec{\alpha})$ of length L, there corresponds a path called its *reverse path* of equal length and labelled $\gamma^{-1} \equiv (m, \vec{\beta})$. The origin of the reverse path m is the endpoint of the original path and vice versa. The path sequence is the reversely ordered one ($\beta_i = \bar{\alpha}_{L-i+1}$). We will also introduce a path composition operation. Given a path $\gamma \equiv (n, \vec{\alpha})$ whose endpoint is m, and another path $\gamma' \equiv (m, \vec{\beta})$, we can construct the composed path $\gamma \circ \gamma' \equiv (n, \vec{\alpha}, \vec{\beta})$.

Now we will give some more definitions.

Definition 2. *A path $\gamma \equiv (n, \alpha_1, \ldots, \alpha_L)$ has* spikes *if there exists one integer i ($1 \leqslant i \leqslant (L-1)$) such that $\alpha_{i+1} = \bar{\alpha}_i$. In the opposite case one says that the path is* terse *or has* no spikes. *The set of all paths without spikes (terse) of length L and origin n is labelled $\bar{\mathcal{S}}_L(n)$ ($\bar{\mathcal{S}}_L(n \to m)$ if the endpoint is fixed to m).*

It is not difficult to obtain $\bar{N}(L)$: the number of elements of $\bar{\mathcal{S}}_L(n)$. Its value is $2d(2d-1)^{L-1}$ for $L \geqslant 1$. The path of zero length is terse $\bar{N}(0) = 1$.

Now we will classify the set of paths into subsets labelled by a terse path. Let us first present the results.

- *There exists a projection $\pi : \mathcal{S}(n \to m) \longrightarrow \bar{\mathcal{S}}(n \to m) \subset \mathcal{S}(n \to m)$ such that to every path γ it associates a terse path $\pi(\gamma)$, called its* reduced path. *If the length of γ is L, then the length of $\pi(\gamma)$ is $L - 2p$, for some integer p.*

Definition 3. *If $\pi(\gamma)$ is the path of length zero, then γ is said to be a* pure spike *path.*

The construction of $\pi(\gamma)$ proceeds iteratively. If the path $\gamma = (n, \vec{\alpha})$ is terse, then $\pi(\gamma) = \gamma$. Otherwise, one can start to scan the sequence of indices i in increasing order, until one finds a value of i such that $\alpha_{i+1} = \bar{\alpha}_i$. This by hypothesis must hold for some i. Then, one can eliminate the elements i and $i+1$ from the sequence, thus defining a new path of length $L-2$. Then, one can apply the procedure once again to the resulting path. In this way, one must proceed iteratively until the iteration terminates. This must necessarily happen since the length of the original path L is finite. The iteration can terminate in two ways. Most frequently, one would reach, at some stage of the iterative procedure, a path without spikes. Then this is precisely $\pi(\gamma)$. In some cases, the iteration proceeds until there are no more elements left in the sequence $\vec{\alpha}$. In this case we would say that the corresponding reduced path is the path of zero length and γ is a pure spike.

The sets $\pi^{-1}(\hat{\gamma})$ will play an important role in our construction. Our main interest is to determine the numbers $N(\hat{\gamma}, p)$: the number of paths of length $2p + \bar{L}$ whose reduced path is $\hat{\gamma}$ (whose length is $\bar{L}$). For the construction we will need to introduce two groups of operations on the sets of paths:

$$\phi : \mathcal{S}_L(n) \longrightarrow \mathcal{S}_{L-1}(n) \tag{1}$$

such that for $\gamma = (n, \alpha_1, \ldots, \alpha_L)$, we have $\phi(\gamma) = (n, \alpha_1, \ldots, \alpha_{L-1})$,

$$\phi_\alpha : \mathcal{S}_L(n) \longrightarrow \mathcal{S}_{L+1}(n). \tag{2}$$

For $\gamma = (n, \alpha_1, \ldots, \alpha_L)$ and $\alpha \in I$, we have $\phi_\alpha(\gamma) = (n, \alpha_1, \ldots, \alpha_L, \alpha)$.

What we need to know is what is the interplay between these operations and the projection π. Let us consider a path $\gamma = (n, \alpha_1, \ldots, \alpha_L)$ whose reduced path is $\pi(\gamma) = (n, \beta_1, \ldots, \beta_{\bar{L}})$. We are interested in the reduced path $\pi(\phi_\alpha(\gamma))$. By the iterative definition of π, we see that after some iterations we would end up with a path $\phi_\alpha(\pi(\gamma))$. Now there can be two cases: if $\bar{\alpha} \neq \beta_{\bar{L}}$ this path is terse and hence $\pi(\phi_\alpha(\gamma)) = (n, \beta_1, \ldots, \beta_{\bar{L}}, \alpha) = \phi_\alpha(\pi(\gamma))$; for the special case $\bar{\alpha} = \beta_{\bar{L}}$, one must still apply one reduction step and the result is $\pi(\phi_{\beta_{\bar{L}}}(\gamma)) = (n, \beta_1, \ldots, \beta_{\bar{L}-1}) = \phi(\pi(\gamma))$.

Now we study $\pi(\phi(\gamma))$. There are again two cases: if $\alpha_L = \beta_{\bar{L}}$ the result is $(n, \beta_1, \ldots, \beta_{\bar{L}-1}) = \phi(\pi(\gamma))$; in the rest of the cases we have $(n, \beta_1, \ldots, \beta_{\bar{L}}, \bar{\alpha}_L) = \phi_{\bar{\alpha}_L}(\pi(\gamma))$. These results can be proven in a similar way as for ϕ_α.

Now we will make use of the previous results. Consider a terse path $\hat{\gamma} = (n, \beta_1, \ldots, \beta_{\bar{L}})$ of non-zero length $\bar{L}$, and consider the set $\mathcal{S}(\hat{\gamma}, p)$ of all paths γ of length $L = \bar{L} + 2p$, with $p \geqslant 1$ an integer, whose reduced path is $\hat{\gamma}$. Then we can conclude:

- *The application ϕ induces a mapping from $\mathcal{S}(\hat{\gamma}, p)$ into $\mathcal{S}(\phi(\hat{\gamma}), p) \cup_{\alpha \neq \beta_{\bar{L}}} \mathcal{S}(\phi_{\bar{\alpha}}(\hat{\gamma}), p - 1)$, which is bijective.*
- *Henceforth, the number of paths $N(\hat{\gamma}, p)$ in $\mathcal{S}(\hat{\gamma}, p)$ satisfies*

$$N(\hat{\gamma}, p) = N(\phi(\hat{\gamma}), p) + \sum_{\alpha \neq \beta_{\bar{L}}} N(\phi_{\bar{\alpha}}(\hat{\gamma}), p - 1).$$

Actually, the number $N(\hat{\gamma}, p)$ does not depend on the path $\hat{\gamma}$ but only on its length $\bar{L}$. We thus conclude

$$N(\bar{L}, p) = N(\bar{L} - 1, p) + (2d - 1)N(\bar{L} + 1, p - 1). \tag{3}$$

- *For a pure spike path $(n, \alpha_1, \ldots, \alpha_L)$, we might apply ϕ and produce a path of length $L - 1$ with reduced path $(n, \bar{\alpha}_L)$. This is also bijective and leads to*

$$N(0, p) = 2dN(1, p - 1). \tag{4}$$

The proof of the previous statements is as follows. The bijectivity can be shown by the existence of an inverse transformation. This is basically ϕ_α with α chosen appropriately. To prove that $N(\hat{\gamma}, p)$ only depends on the length can be done by induction. Prove directly by construction that the statement is true for paths of short length (it is easy to solve the problem up to $L = 4$, for example). Then one assumes that the statement is verified up to length $L = \bar{L} + 2p$ (for any p). Then one can use formulae (3) and (4) to show that the statement is true for paths of length $L + 1$. Notice that if the right-hand side does not depend on the actual terse paths but only on its lengths, and since these lengths only depend on the length of the reduced path $\hat{\gamma}$ of the left-hand side, the result follows.

By repeated application of the relations equation (3) and (4), together with the initial condition $N(\bar{L}, 0) = 1$, one can obtain all the $N(\bar{L}, p)$ values. To exploit these relations we will introduce the following generating functions:

$$F(\bar{L}, z) = \sum_{p=0}^{\infty} z^p N(\bar{L}, p) \tag{5}$$

$$G(y, z) = \sum_{\bar{L}=0}^{\infty} F(\bar{L}, z)\, y^{\bar{L}}. \tag{6}$$

Multiplying relation (3) by the appropriate powers of z and y and summing over p and $\bar{L}$, one finds

$$G(y, z) - F(0, z) = yG(y, z) + \frac{(2d - 1)z}{y}(G(y, z) - F(0, z) - yF(1, z)) \tag{7}$$

and from it, one can write G in terms of F:

$$G(y, z) = \frac{1}{(y(1 - y) - (2d - 1)z)}\left(\left(\frac{y}{2d} - (2d - 1)z\right)F(0, z) + y\frac{2d - 1}{d}\right). \tag{8}$$

Notice that the zeros of the denominator in the previous expression can give rise to singularities, even for small values of z and y, unless the numerator vanishes at these zeros. This must actually happen since G and F can be shown to be analytic in a neighbourhood of $y = z = 0$ (this follows from $N(\bar{L}, p) < (2d)^{\bar{L}+2p}$). This allows one to determine $F(0, z)$:

$$F(0, z) = \frac{2d-1}{d} \frac{1}{1 + [d/(d-1)]\sqrt{1-4(2d-1)z}}. \tag{9}$$

Now plugging this expression into (8) we obtain the formula for $G(y, z)$.

The expressions can be simplified with a suitable change of variables. Let us introduce the variable ξ:

$$\xi(z) = \tfrac{1}{2}\left(1 - \sqrt{1-4(2d-1)z}\right) \tag{10}$$

with inverse

$$z(\xi) = \frac{\xi(1-\xi)}{2d-1}. \tag{11}$$

Then one can conclude:

$$F(0, z(\xi)) = \frac{1}{1 - 2d\xi/(2d-1)} \tag{12}$$

$$G(y, z(\xi)) = -\frac{(2d-1)(1-\xi)}{2d-1-2d\xi} \frac{1}{y+\xi-1}. \tag{13}$$

Notice that the only dependence on y sits in the last denominator. It is now fairly simple to obtain $F(\bar{L}, z)$ by picking the relevant power of y in the expansion. One finds

$$F(\bar{L}, z(\xi)) = \frac{1}{(1-(2d/(2d-1))\xi)} \frac{1}{(1-\xi)^{\bar{L}}}. \tag{14}$$

The last formula is the main one of this section. From it one can obtain the numbers $N(\bar{L}, p)$, by differentiation or Cauchy integration. This we will do later.

Before that, as a check of our formulae, one can compute the number of paths of length L as a sum over the number of terse paths times the number of paths of length L having a given terse path as a reduced path:

$$(2d)^L = N(L) = \sum_{p=0}^{[L/2]} \bar{N}(L-2p)N(L-2p, p). \tag{15}$$

To check all formulae at the same time we can multiply the expression by z^L and sum over L. We obtain

$$\begin{aligned}
\frac{1}{1-2dz} &= F(0, z^2) + \sum_{\bar{L}=1}^{\infty} z^{\bar{L}} F(\bar{L}, z^2) 2d(2d-1)^{\bar{L}-1} \\
&= \frac{1}{(1-(2d/(2d-1))\xi(z^2))}\left(1 + \frac{2dz}{1-\xi(z^2)-(2d-1)z}\right) \\
&= \frac{1-\xi(z^2)+z}{(1-(2d/(2d-1))\xi(z^2))(1-\xi(z^2)-(2d-1)z)}.
\end{aligned} \tag{16}$$

For the second identity of the previous formula we have resummed a geometric series. Finally, to prove that the right-hand side of the previous equation coincides with the left-hand side, one must simply manipulate algebraically the expression and use the relation $\xi(z^2)(1-\xi(z^2)) = (2d-1)z^2$.

We conclude this section by extracting the numbers $N(L, p)$ themselves. This can be done by employing the expression of the generating function (14), and making a contour integral in the complex plane of z around the origin, and using Cauchy's theorem. It is more practical to change variables from z to ξ in the integral. Notice that for $|z|$ small enough, the contour in ξ also encircles the origin and the function $F(L, z(\xi))$ has no singularities inside. The resulting integrand is a product of negative powers of ξ and of $(1-\xi)$, times the factor $1/(1-(2d\xi/(2d-1)))$ coming from (14). If one expands this denominator in powers of ξ, it is not hard to show that

$$N(L, p) = \sum_{j=0}^{p} (2d)^j (2d-1)^{p-j} \frac{(L+2p-j-1)!}{(p-j)!(L+p)!}(L+j). \tag{17}$$

We see that the resulting expression is a polynomial in d of degree p. To obtain the coefficients of the different powers of d, one could expand the power of $(2d-1)$ in powers of d and rearrange the summation. The method can be carried out but is fairly lengthy and complicated. A short cut to arrive at the same final expression is to multiply and divide equation (17) by $j!$. Then one replaces the factors $j!$ and $(L+2p-j-1)!$ by their standard integral representation (that of Euler's gamma function) and performs the sum over j. The expression is then given as a double integral over two variables α and β going from 0 to ∞. Now one can perform the standard trick in computing Feynman integrals by changing variables to $\lambda \equiv (\alpha+\beta)$ and the *Feynman parameter* $x \equiv \alpha/\lambda$. The integration over λ can be performed and we arrive at

$$\begin{aligned} N(L, p) &= \binom{L+2p}{p} \int_0^1 \mathrm{d}x\, x^{L+p-1}(2d-x)^{p-1}(L(2d-x)+2dp(1-x)) \\ &= \binom{L+2p}{p} \sum_{s=0}^{p} \frac{(2d)^s(-1)^{p-s}}{(L+2p-s)} \binom{p}{s} \left(L + \frac{s}{L+2p-s+1}\right). \end{aligned} \tag{18}$$

The first equality in the previous expression is a Feynman parameter integral representation of the numbers $N(L, p)$. The second one is a representation as a polynomial in d, and it can be obtained easily from the other. We have given the expression for $N(L, p)$ for completeness, though in actual applications it is more useful to work with the generating function $F(L, z)$.

3. Summing over reduced paths

In this section we will compute a matrix generating function for the set of terse paths $\bar{\mathcal{S}}_L(n)$. This generating function turns out to be useful in applications to strong-coupling expansions of lattice models. Let us introduce a collection of matrices $\boldsymbol{A}_\alpha$ for $\alpha \in I$. The interesting quantity to study is

$$\mathcal{T}(\boldsymbol{A}) = \sum_{L=0}^{\infty} \sum_{(n,\vec{\alpha}) \in \bar{\mathcal{S}}_L(n)} \boldsymbol{A}_{\alpha_1} \ldots \boldsymbol{A}_{\alpha_L}. \tag{19}$$

We will first compute $\mathcal{T}(\boldsymbol{A})$ for matrices $\boldsymbol{A}$ satisfying $\boldsymbol{A}_\alpha \boldsymbol{A}_{\bar{\alpha}} = \lambda \boldsymbol{I}$ (i.e. a multiple of the identity). This condition is satisfied in some of the most important applications of the formula. At the end of the section we will give the more general formulae.

To facilitate the reading of the following for those who are mostly interested in the result, we will begin by giving the answer:

$$\mathcal{T}(\boldsymbol{A}) = (1-\lambda)(1+(2d-1)\lambda - \tilde{\boldsymbol{A}})^{-1} \tag{20}$$

with

$$\tilde{\boldsymbol{A}} = \sum_{\alpha \in I} \boldsymbol{A}_\alpha. \tag{21}$$

The conditions on the matrices $\boldsymbol{A}$ for which the previous expression applies can be read out from it. The eigenvalues of $\tilde{\boldsymbol{A}}$ must be small enough for the inverse matrix entering in equation (20) to exist. In the following paragraphs we will give the proof of this result.

We begin by considering the set of all terse paths, with origin in n, length $L \geqslant 1$ and ending with step α: $\bar{\mathcal{S}}_L^{\alpha}(n)$. Applying a similar definition to equation (19) to this set we obtain

$$\mathcal{T}_\alpha(L, \boldsymbol{A}) = \sum_{\gamma \in \bar{\mathcal{S}}_L^{\alpha}(n)} \boldsymbol{A}_{\alpha_1} \cdots \boldsymbol{A}_{\alpha}. \tag{22}$$

Now, clearly the path $\phi(\gamma)$ is also a terse path and has length $L-1$, but it cannot end with step $\bar{\alpha}$. Hence,

$$\mathcal{T}_\alpha(L+1, \boldsymbol{A}) = \sum_{\beta \neq \bar{\alpha}} \mathcal{T}_\beta(L, \boldsymbol{A}) \boldsymbol{A}_\alpha. \tag{23}$$

The formula is valid for $L \geqslant 1$.

Now from it we will derive an equation for $\mathcal{T}_\alpha(\boldsymbol{A}) \equiv \sum_{L=1}^{\infty} \mathcal{T}_\alpha(L, \boldsymbol{A})$. Then our main quantity $\mathcal{T}(\boldsymbol{A})$ is given by $\boldsymbol{I} + \sum_{\alpha \in I} \mathcal{T}_\alpha(\boldsymbol{A})$. Summing equation (23) over L one obtains

$$\mathcal{T}_\alpha(\boldsymbol{A}) = (\mathcal{T}(\boldsymbol{A}) - \mathcal{T}_{\bar{\alpha}}(\boldsymbol{A}))\boldsymbol{A}_\alpha. \tag{24}$$

Given $\mathcal{T}(\boldsymbol{A})$, these are coupled equations for the indices α and $\bar{\alpha}$. We then write them as a single vector equation:

$$(\mathcal{T}_\alpha(\boldsymbol{A}), \mathcal{T}_{\bar{\alpha}}(\boldsymbol{A}))\boldsymbol{\mathcal{H}} = \mathcal{T}(\boldsymbol{A})\,(\boldsymbol{A}_\alpha, \boldsymbol{A}_{\bar{\alpha}}) \tag{25}$$

where $\boldsymbol{\mathcal{H}}$ is an invertible matrix. This matrix and its inverse are given by the formulae

$$\boldsymbol{\mathcal{H}} = \begin{pmatrix} \boldsymbol{I} & \boldsymbol{A}_{\bar{\alpha}} \\ \boldsymbol{A}_\alpha & \boldsymbol{I} \end{pmatrix} \tag{26}$$

$$\boldsymbol{\mathcal{H}}^{-1} = \frac{1}{(1-\lambda)} \begin{pmatrix} \boldsymbol{I} & -\boldsymbol{A}_{\bar{\alpha}} \\ -\boldsymbol{A}_\alpha & \boldsymbol{I} \end{pmatrix}. \tag{27}$$

Then, for fixed $\mathcal{T}(\boldsymbol{A})$, one can solve for $\mathcal{T}_\alpha(\boldsymbol{A})$, obtaining

$$\mathcal{T}_\alpha(\boldsymbol{A}) = \frac{\mathcal{T}(\boldsymbol{A})}{(1-\lambda)}(\boldsymbol{A}_\alpha - \lambda). \tag{28}$$

Finally, summing both sides of the equation over α we find

$$\mathcal{T}(\boldsymbol{A}) - \boldsymbol{I} = \frac{\mathcal{T}(\boldsymbol{A})}{(1-\lambda)}(\tilde{\boldsymbol{A}} - 2d\lambda). \tag{29}$$

From this equation one can solve for $\mathcal{T}(\boldsymbol{A})$ obtaining equation (20).

We can now, as in the previous section, check the formula by using it in deriving a known result. Consider the sum over all paths $\gamma = (n, \vec{\alpha})$ of the ordered product of the matrices $\boldsymbol{A}_\alpha$. Since this is a geometric series it is easily resummed to $(\boldsymbol{I} - \tilde{\boldsymbol{A}})^{-1}$. Now this result has to be reobtained by splitting the sum over paths into a sum over terse paths and a sum over paths whose reduced path is a given terse path. In this way one is making use of the results of the last section and of this section at the same time. The last summation can be performed in terms of the generating function studied in the last section. One has

$$\sum_{L=0}^{\infty} \sum_{(n,\vec{\alpha}) \in \bar{\mathcal{S}}_L(n)} \boldsymbol{A}_{\alpha_1} \cdots \boldsymbol{A}_{\alpha_L} F(L, \lambda). \tag{30}$$

Now, given the form of $F(L, \lambda)$ given in expression (14), one recognizes the structure given in equation (19) with $\boldsymbol{A}_\alpha$ rescaled. We find

$$\frac{1}{(1-(2d/(2d-1))\xi(\lambda))}\,\mathcal{T}(\boldsymbol{A}/(1-\xi(\lambda))). \tag{31}$$

Now using our expression for $\mathcal{T}(\boldsymbol{A})$ (equation (20)) and the relation between $\xi(\lambda)$ and λ (equation (10)) one obtains the known result.

Now, as announced at the beginning of the section we will give the result without imposing the condition $\boldsymbol{A}_\alpha \boldsymbol{A}_{\bar{\alpha}} = \lambda \boldsymbol{I}$. It is not difficult by following the same steps as before to show that in the general case we have

$$\mathcal{T}(\boldsymbol{A}) = (1-\tilde{\boldsymbol{B}})^{-1} \tag{32}$$

$$\mathcal{T}_\alpha(\boldsymbol{A}) = \mathcal{T}(\boldsymbol{A})\,(\boldsymbol{A}_\alpha - \boldsymbol{A}_{\bar{\alpha}}\boldsymbol{A}_\alpha)\,(\boldsymbol{I} - \boldsymbol{A}_{\bar{\alpha}}\boldsymbol{A}_\alpha)^{-1} \tag{33}$$

where

$$\tilde{\boldsymbol{B}} = \sum_\alpha (\boldsymbol{A}_\alpha - \boldsymbol{A}_{\bar{\alpha}}\boldsymbol{A}_\alpha)\,(\boldsymbol{I} - \boldsymbol{A}_{\bar{\alpha}}\boldsymbol{A}_\alpha)^{-1}. \tag{34}$$

The previously given formula (20) follows as a special case from this one.

To conclude this section, we comment that, using the above formulae, one can derive matrix generating functions for the sum of terse paths with fixed origin and endpoint. For that purpose one simply has to multiply $\boldsymbol{A}_\alpha$ by a phase $\mathrm{e}^{\mathrm{i}\varphi_\alpha}$. For the reverse direction $\bar{\alpha}$ one has the complex conjugate phase ($\varphi_{\bar{\alpha}} = -\varphi_\alpha$), so that the condition $\boldsymbol{A}_\alpha \boldsymbol{A}_{\bar{\alpha}} = \lambda \boldsymbol{I}$ is respected. With a suitable integration over the phases φ_μ one can restrict the sum over terse paths to those having a fixed endpoint. For example, if one wants to evaluate the contribution to $\mathcal{T}(\boldsymbol{A})$ from paths whose origin is n and endpoint is m one can simply write

$$\prod_\mu \left(\int_0^{2\pi} \frac{\mathrm{d}\varphi_\mu}{2\pi}\, \mathrm{e}^{\mathrm{i}\varphi_\mu (n_\mu - m_\mu)} \right) \mathcal{T}(\mathrm{e}^{\mathrm{i}\varphi_\alpha} \boldsymbol{A}_\alpha). \tag{35}$$

4. Closed and simple paths

In this section we will look at closed paths: a path such that its origin and endpoint coincide. One frequently encounters situations in which closed paths play an important role, such as in computing the free energy or the fermion determinant. In those cases, for every such path one has to evaluate a trace. For that purpose, the notion of terse paths is somehow insufficient. We would like to single out those paths $(n, \vec{\alpha})$ for which the last step α_L differs from $\bar{\alpha}_1$. We will call those paths simple. Let us now give the definition more precisely.

Definition 4. *A path $\gamma = (n, \vec{\alpha})$ of length L is* simple, *if it is terse (without spikes) and in addition one has $\alpha_L \neq \bar{\alpha}_1$.*

Simple paths can be open or closed, however, their usefulness appears normally when they are closed. The set of simple paths of length L and origin in n is labelled $\tilde{\mathcal{S}}_L(n)$. The set $\tilde{\mathcal{S}}_0(n)$ is given by the path of zero length.

By a similar procedure to the one followed in section 1, one can associate to any path γ a given simple path $\tilde{\pi}(\gamma)$. To construct $\tilde{\pi}(\gamma)$ one starts by obtaining the reduced path $(\pi(\gamma) = \hat{\gamma} \equiv (n, \vec{\beta}))$ associated to γ. Let us consider that its length is L and its origin is n. If this terse path is simple, then this is precisely $\tilde{\pi}(\gamma)$. If not, it is due to β_L being equal to $\bar{\beta}_1$. Hence, we eliminate the first and last steps in the sequence (β_1 and β_L). The resulting path is terse and has length $L-2$. Notice, however, that its origin is now $n + V(\beta_1)$, and not n. If the path is simple, then it coincides with $\tilde{\pi}(\gamma)$, otherwise one has to repeat the procedure once more. Eventually, one reaches a simple path, which could be just a path of zero length.

Our first goal is to develop similar counting rules for simple paths to those obtained in section 2 for terse paths. In particular, we are interested in computing the numbers $\widetilde{N}(\tilde{\gamma}, p)$: the number of paths γ of length $l + 2p$ whose associated simple path is $\tilde{\pi}(\gamma) = \tilde{\gamma}$ (of length l). The generating function of these numbers is

$$\widetilde{F}(\tilde{\gamma}, z) = \sum_{p=0}^{\infty} z^p \widetilde{N}(\tilde{\gamma}, p). \tag{36}$$

In what follows we will compute this generating function and the numbers $\widetilde{N}(\tilde{\gamma}, p)$.

The procedure that we will employ is to relate these numbers with $N(l, p)$. For that purpose consider a path γ with $\tilde{\pi}(\gamma) = \tilde{\gamma} \equiv (m, \vec{\alpha})$ and consider its reduced path $\pi(\gamma) \equiv (n, \vec{\beta})$. It is clear from the description of the construction of $\tilde{\pi}(\gamma)$ that the path $\pi(\gamma)$ must be the composition of three paths:

$$\pi(\gamma) = s \circ \tilde{\gamma} \circ s^{-1}.$$

The path $s \equiv (n, \vec{\rho})$ is a terse path of length p' ($0 \leqslant p' \leqslant p$) going from n to m. If the path $\tilde{\gamma}$ has length l then $\rho_{p'} \neq \bar{\alpha}_1, \alpha_l$. This must hold, since the composition $s \circ \tilde{\gamma} \circ s^{-1}$ must be terse. Conversely all paths γ having $\pi(\gamma) = s \circ \tilde{\gamma} \circ s^{-1}$ have $\tilde{\gamma}$ as its associated simple path. Hence, all one needs to do is to count for each case of $\pi(\gamma)$ the number of paths γ of length $l + 2p$. The main formula is

$$\widetilde{N}(\tilde{\gamma}, p) = N(l, p) + \sum_{p'=1}^{p} N(l + 2p', p - p')\,(2d - 2)(2d - 1)^{p'-1}. \tag{37}$$

The quantity $(2d - 2)(2d - 1)^{p'-1}$ counts the number of acceptable terse paths s of length p' going from any point n to m. The word acceptable refers to the condition $\rho_{p'} \neq \bar{\alpha}_1, \alpha_l$. The previous formula (37) is valid for l and p strictly positive. If any of the two is zero then $\widetilde{N} = N$.

A first conclusion from formula (37) is that $\widetilde{N}(\tilde{\gamma}, p)$ only depends on the length l of the simple path $\tilde{\gamma}$. If we multiply both sides of the equation by z^p and sum over p, we find (for $l > 0$)

$$\widetilde{F}(l, z) = F(l, z) + \sum_{p'=1}^{\infty} (2d - 2)(2d - 1)^{p'-1} z^{p'}. \tag{38}$$

Now using the form of $F(l, z)$ one obtains

$$\widetilde{F}(l, z) = \frac{1}{1 - 2\xi(z)} \frac{1}{(1 - \xi(z))^l}. \tag{39}$$

This formula is valid for $l > 0$. This is complemented by $\widetilde{F}(0, z) = F(0, z)$. To extract from $\widetilde{F}(l, z)$ the numbers $\widetilde{N}(l, p)$, one proceeds as before by Cauchy integration. The calculation is now much simpler since $(1 - 2\xi(z))$ is up to a constant the Jacobian for the change of variables from z to ξ. Finally, one obtains ($l > 0$)

$$\widetilde{N}(l, p) = \frac{(2d - 1)^p (2p + l)!}{p!(p + l)!}. \tag{40}$$

In some applications, one is interested in a slight variant of the generating function $\widetilde{F}(l,z)$. Its definition and final expression is given by

$$\widetilde{F}'(l,z) \equiv \sum_{p=0}^{\infty} \frac{1}{l+2p} \widetilde{N}(l,p) z^p = \frac{1}{l(1-\xi(z))^l}. \tag{41}$$

The last expression, valid for positive l, could be obtained after some work by integration of $\widetilde{F}(l,z)$. The $\frac{1}{l+2p}$ in the definition of $\widetilde{F}'$ occurs naturally when the sum of closed paths is the result of a fermionic or bosonic determinant. We complement this result with the one for $l=0$:

$$\widetilde{F}'(0,z) \equiv \sum_{p=0}^{\infty} \frac{1}{2p} N(0,p) z^p = d\log(1-\xi(z)) + (d-1)\log\left(1 - \frac{2d}{2d-1}\xi(z)\right). \tag{42}$$

The remaining part of this section is dedicated to the evaluation of sums over simple paths. The basic quantity is

$$\widetilde{\mathcal{T}}(\boldsymbol{A}) = \sum_{\tilde{L}=0}^{\infty} \sum_{(n,\vec{\alpha})\in\tilde{\mathcal{S}}_L(n)} \boldsymbol{A}_{\alpha_1}\cdots\boldsymbol{A}_{\alpha_{\tilde{L}}}. \tag{43}$$

where $\tilde{\mathcal{S}}_L(n)$ is the set of all simple paths with origin in n and length L. In short, what we want is the generalization of the quantity defined in equation (19) but restricted to simple closed paths. Similarly to what we did in section 3, we will first present the result, and then give the derivation. We obtain

$$\widetilde{\mathcal{T}}(\boldsymbol{A}) = \frac{1}{(1-\lambda)}(2\lambda(d-1) + (1+\lambda^2(2d-1))\boldsymbol{H} - \sum_{\alpha\in I} \boldsymbol{A}_\alpha \boldsymbol{H} \boldsymbol{A}_{\bar{\alpha}}) \tag{44}$$

with

$$\boldsymbol{H} = (1 + (2d-1)\lambda - \tilde{\boldsymbol{A}})^{-1} \tag{45}$$

where $\boldsymbol{A}_\alpha \boldsymbol{A}_{\bar{\alpha}} = \lambda \boldsymbol{I}$ as in section 3.

The derivation follows a similar track to the one employed for $\mathcal{T}(\boldsymbol{A})$. Our first goal is the calculation of the quantity $\mathcal{T}_{\alpha\alpha'}(L,\boldsymbol{A})$ given by

$$\mathcal{T}_{\alpha\alpha'}(L,\boldsymbol{A}) = \sum_{(n,\vec{\alpha})\in\tilde{\mathcal{S}}_L^{\alpha\alpha'}(n)} \boldsymbol{A}_{\alpha_1}\cdots\boldsymbol{A}_{\alpha_L} \tag{46}$$

where $\tilde{\mathcal{S}}_L^{\alpha\alpha'}(n)$ is the set of all simple paths $(n,\vec{\alpha})$ of length L ($L>2$) and origin in n such that $\alpha_1 = \alpha$ and $\alpha_L = \alpha'$. The main iteration equation allowing the evaluation of this quantity is

$$\mathcal{T}_{\alpha\alpha'}(L+2,\boldsymbol{A}) = \sum_{\alpha\neq\bar{\beta};\alpha'\neq\bar{\beta}'} \boldsymbol{A}_\alpha \mathcal{T}_{\beta\beta'}(L,\boldsymbol{A}) \boldsymbol{A}_{\alpha'}. \tag{47}$$

The sum of $\mathcal{T}_{\alpha\alpha'}(L,\boldsymbol{A})$ over L ranging from 2 to ∞ is denoted by $\mathcal{T}_{\alpha\alpha'}(\boldsymbol{A})$. An equation for this quantity follows from summing both sides of equation (47) over L. After similar manipulations as those of section 3, one finds

$$\mathcal{T}_{\alpha\alpha'}(\boldsymbol{A}) = \boldsymbol{A}_\alpha \mathcal{T}_{\bar{\alpha}\bar{\alpha}'}(\boldsymbol{A}) \boldsymbol{A}_{\alpha'} + S_{\alpha\alpha'} \tag{48}$$

with

$$S_{\alpha\alpha'} = -\delta_{\alpha\bar{\alpha}'}\lambda + \lambda\delta_{\alpha\alpha'}\boldsymbol{A}_\alpha + (1+\lambda)\boldsymbol{A}_\alpha\boldsymbol{H}\boldsymbol{A}_{\alpha'} + \lambda(\boldsymbol{H}\boldsymbol{A}_{\alpha'} + \boldsymbol{A}_\alpha\boldsymbol{H})$$

where $\boldsymbol{H}$ is the quantity defined in equation (45). Finally, combining the equation for $\mathcal{T}_{\alpha\alpha'}(\boldsymbol{A})$ and for $\mathcal{T}_{\bar{\alpha}\bar{\alpha}'}(\boldsymbol{A})$, one can solve for $\mathcal{T}_{\alpha\alpha'}(\boldsymbol{A})$:

$$\mathcal{T}_{\alpha\alpha'}(\boldsymbol{A}) = \frac{1}{(1-\lambda)}(\lambda(-\delta_{\alpha\bar{\alpha}'} + \delta_{\alpha\alpha'}\boldsymbol{A}_\alpha - \boldsymbol{H}\boldsymbol{A}_{\alpha'} - \boldsymbol{A}_\alpha\boldsymbol{H}) + \boldsymbol{A}_\alpha\boldsymbol{H}\boldsymbol{A}_{\alpha'} + \lambda^2\boldsymbol{H}). \tag{49}$$

The previous quantity can be related to $\widetilde{\mathcal{T}}(\boldsymbol{A})$ as follows:

$$\widetilde{\mathcal{T}}(\boldsymbol{A}) = \boldsymbol{I} + \widetilde{\boldsymbol{A}} + \sum_{\alpha\neq\bar{\alpha}'} \mathcal{T}_{\alpha\alpha'}(\boldsymbol{A}). \tag{50}$$

Using this result in combination with equation (49) one obtains the final formula (44). One can again check the validity of the expression by using it in reobtaining a known result. We leave this to the reader. We recall that in the definition of $\widetilde{\mathcal{T}}(\boldsymbol{A})$ one sums over all simple paths, closed or open. Restricting oneself to closed paths can be done with the same technique explained at the end of the previous section.

5. Discussion

In this section we will exemplify how to apply the previous results to some physical problems. We consider a lattice model involving continuous spin variables with nearest-neighbour interactions. These lattice fields can be real or complex valued or Grassman variables if they describe fermions. To apply the path representation we need a quadratic action or Hamiltonian in these fields. For example, for complex fields one has

$$\sum_{a,b,n,m} \phi^a(n)^\dagger \phi^b(m)\, M^{ab}(n,m). \tag{51}$$

The indices n, m label lattice points and the indices a, b are internal. The matrix M will depend on other fields. For instance, in many cases constraints or non-quadratic terms in the lattice action can be rewritten as a quadratic (Gaussian) Hamiltonian with the aid of auxiliary fields. Then, one can integrate out these complex fields ($\phi^a(n)$) using the Gaussian integration formulae. The two quantities entering the final expressions are the determinant of M (det M) and the inverse of M. Now, the nearest-neighbour character of our matrix manifests itself in that we can write (after an adequate rescaling of the fields if necessary)

$$M = \boldsymbol{I} - \sum_{\alpha\in I} \Delta_\alpha \tag{52}$$

where the matrix Δ_α can be written as

$$\Delta_\alpha^{ab}(n,m) = \boldsymbol{A}_\alpha^{ab}(n)\, \delta_{mn+V(\alpha)}. \tag{53}$$

It only produces transitions between a lattice point n and its neighbour in the α direction $n + V(\alpha)$. It is this form of the matrix M that allows a random walk (lattice path) representation of the determinant or the inverse of M. Our formulae allow a rearrangement of this sum over paths into a sum over simple closed paths or terse paths, respectively. This is feasible whenever

$$\Delta_\alpha \Delta_{\bar{\alpha}} = \Lambda \tag{54}$$

with Λ a matrix which is independent of the spacetime point. This occurs naturally whenever the matrix Δ_α, although dependent on the spacetime point, involves unitary link fields as in $U(N)$ or Z_N gauge theories. We will restrict to the case when Λ is a multiple of the identity $\lambda\boldsymbol{I}$.

Now if we denote by $\boldsymbol{A}(\gamma)$ the ordered product of the matrices $\boldsymbol{A}_\alpha(n)$ along the path γ, we can write

$$\log(\det(M))/\mathcal{V} = \left(-d\log(1-\xi(\lambda)) + (d-1)\log\left(1 - \frac{2d}{2d-1}\xi(\lambda)\right)\right)\mathrm{Tr}(\boldsymbol{I}) + \sum_{l=1}^{\infty}\frac{1}{l}\sum_{\tilde{\gamma}\in\tilde{\mathcal{S}}_l(n\to n)}\frac{\mathrm{Tr}(A(\tilde{\gamma}))}{(1-\xi)^l} \tag{55}$$

where $\mathcal{V}$ is the lattice volume and $\xi(\lambda)$ is defined in equation (10). To arrive at the previous equation, we have rearranged as usual the sum over paths into a sum over simple paths, and used the results of the previous sections. The term proportional to $\mathrm{Tr}(\boldsymbol{I})$, equal to $\widetilde{F}'(0,\lambda)$, gives the contribution of pure spike paths. In some theories, like $U(N)$ gauge theories at strong coupling in the large-N limit with either bosonic or fermionic spin fields in either the fundamental or adjoint representation, this term turns out to be the only surviving one [8]. Thus, up to a multiplicative constant depending on the type of field, $\widetilde{F}'(0,\lambda)$ (equation (42)) gives the free energy per unit volume in that limit. We suggest that in other theories, the rearrangement into simple paths could be an effective method to perform the summation over paths.

Now as an additional application, let us compute the pure spike contribution to the *mesonic effective potential*. Let us add to the action (51) a mesonic source term:

$$-\sum_{n,a,b}\phi^a(n)^\dagger\phi^b(n)\,J^{ab}(n) \tag{56}$$

where $J(n)$ acts as the source of local field bilinears (mesons). Integrating over the Gaussian fields ϕ we obtain the connected generating functional $W(J)$:

$$W(J) \equiv \log(Z(J)/Z(0)) = \sum_{k=1}^{\infty}\frac{1}{k}\mathrm{Tr}((M^{-1}J)^k) \tag{57}$$

where the trace includes a summation over lattice points. Each factor of M^{-1} can be expanded into a sum over paths (random walks). The pure spike contribution $W_0(J)$ is that in which the overall path obtained within each trace is a pure spike path. Again this contribution would be the leading one if the matrices $\boldsymbol{A}_\alpha(n)$ entering equation (51) involve random $U(N)$ fields at large N. In the subsequent expressions only the remaining part of the $\boldsymbol{A}_\alpha$ would enter, which we will take to be constant in what follows. In order to implement the restriction to pure spike paths, it is convenient to express the propagators M^{-1} as a sum over terse paths:

$$(M^{-1}(n,m))^{ab} = \frac{1}{(1-(2d/(2d-1))\xi)}\sum_{l=0}^{\infty}\sum_{\hat{\gamma}\in\bar{\mathcal{S}}_l(n\to m)}\frac{(A(\hat{\gamma}))^{ab}}{(1-\xi)^l}. \tag{58}$$

We then obtain for $W_0(J)$:

$$W_0(J) = \sum_{k=1}^{\infty}\frac{1}{k}\sum_{x_1,\ldots,x_k\in\mathcal{L}}\ \sum_{\hat{\gamma}_1\in\bar{\mathcal{S}}(x_1\to x_2)}\cdots\sum_{\hat{\gamma}_k\in\bar{\mathcal{S}}(x_k\to x_1)} \times \mathrm{Tr}(J'(x_1)A'(\hat{\gamma}_1)J'(x_2)\cdots A'(\hat{\gamma}_k))\,\Theta(\hat{\gamma}_1\circ\hat{\gamma}_2\ldots\circ\hat{\gamma}_k) \tag{59}$$

where $\Theta(\gamma)$ is 1 if γ is a pure spike path and zero otherwise, and the rescaled quantities $\boldsymbol{A}'$, J' are given by

$$\boldsymbol{A}'_\alpha = \frac{\boldsymbol{A}_\alpha}{(1-\xi(\lambda))} \tag{60}$$

$$J'(n) = \frac{J(n)}{(1-(2d/(2d-1))\xi(\lambda))}. \tag{61}$$

We see that the net effect of replacing the sum over paths by a sum over terse paths is precisely this rescaling, as follows from our results of section 2. The first two terms of $W_0(J)$ are

$$W_0(J) = \sum_{x\in\mathcal{L}} \mathrm{Tr}(J'(x)) + \tfrac{1}{2} \sum_{x_1,x_2\in\mathcal{L}} \overline{J}'(x_1)\mathcal{P}(x_1 \to x_2)J'(x_2) + \cdots . \tag{62}$$

The linear term in J' is trivial since the only path that contributes is the path of zero length. The constraint $\Theta(\hat{\gamma}_1 \circ \hat{\gamma}_2)$ for the quadratic term implies that $\hat{\gamma}_2$ must be the reverse path of $\hat{\gamma}_1$. The resummation over terse paths can be done with the use of the formulae of section 3. One obtains the following explicit expression of the propagator $\mathcal{P}(x_1 \to x_2)$:

$$\mathcal{P}(x_1 \to x_2) = \prod_\mu \left(\int_0^{2\pi} \frac{\mathrm{d}\varphi_\mu}{2\pi}\, \mathrm{e}^{\mathrm{i}\varphi_\mu (x_1 - x_2)_\mu} \right) (1-\lambda')\,(1-(2d-1)\lambda' - \boldsymbol{B})^{-1} \tag{63}$$

where

$$\lambda' = \frac{\lambda^2}{(1-\xi(\lambda))^4} \tag{64}$$

$$\boldsymbol{B} = \sum_{\alpha\in I} \mathrm{e}^{\mathrm{i}\varphi_\alpha} \boldsymbol{A}'_\alpha \otimes (\boldsymbol{A}'_{\bar{\alpha}})^t. \tag{65}$$

These expressions were used in our recent paper on $N = 1$ SUSY Yang–Mills [7]. In formula (62) $J'(n)$ has to be looked at as a column vector on which the matrix $\mathcal{P}$ acts. Then, $\overline{J}'(x_1)$ is the row vector whose elements are the transpose of J'.

Finally, we will address the calculation of the cubic term in $W_0(J)$. For that purpose we have to solve the constraint $\Theta(\hat{\gamma}_1 \circ \hat{\gamma}_2 \circ \hat{\gamma}_3)$. In the generic case, this can be solved as follows:

$$\hat{\gamma}_1 = s_2^\alpha \circ \left(s_3^\beta\right)^{-1} \tag{66}$$

$$\hat{\gamma}_2 = s_3^\beta \circ \left(s_1^\gamma\right)^{-1} \tag{67}$$

$$\hat{\gamma}_3 = s_1^\gamma \circ \left(s_2^\alpha\right)^{-1} \tag{68}$$

where $s_1^\gamma \in \bar{\mathcal{S}}^\gamma$ is a terse path ending with a step in the γ direction, and similar definitions for s_2 and s_3. Furthermore, one must have $\alpha \neq \beta \neq \gamma \neq \alpha$. The exceptional cases occur when any of the paths s_i is a path of zero length. It is clear that the summation over the paths s_i can be done with the aid of the formulae of section 3. The best way to express the result is in terms of the mean mesonic field $\Phi^{ab}(x)$:

$$\Phi(x) = \sum_{x'\in\mathcal{L}} \mathcal{P}(x \to x')J'(x'). \tag{69}$$

Then the cubic term in $W_0(J)$ becomes

$$\sum_{x\in\mathcal{L}} \Bigg(\tfrac{1}{3}\,\mathrm{Tr}\big(\Phi^3(x)\big) - \sum_{\alpha\in I} \mathrm{Tr}\big(\Phi(x)\Phi_\alpha^2(x)\big) - 2\sum_{\alpha\neq\beta} \mathrm{Tr}\big(\Phi_\beta(x)\Phi_\alpha^2(x)\big)$$
$$- \tfrac{2}{3} \sum_{\alpha\neq\beta\neq\gamma\neq\alpha} \mathrm{Tr}(\Phi_\alpha(x)\Phi_\beta(x)\Phi_\gamma(x)) \Bigg) \tag{70}$$

where

$$\Phi_\alpha(x) = \frac{1}{1-\lambda'} (\boldsymbol{A}'_\alpha \Phi(x + V(\alpha))\boldsymbol{A}'_{\bar{\alpha}} - \lambda'\Phi(x)). \tag{71}$$

This coincides up to a normalization factor with the cubic term in the effective action $\Gamma_0(\Phi)$, which is the Legendre transform of $W_0(J)$. Following a similar procedure one can compute quartic and higher terms in $\Gamma_0(\Phi)$.

Acknowledgments

The author acknowledges useful conversations with E Gabrielli and C Pena. This work has been partly financed by CICYT under grant AEN97-1678.

References

[1] Wilson K G 1977 New phenomena in subnuclear physics *Erice Lectures 1975* ed A Zichichi (New York: Plenum)
[2] Creutz M 1983 *Quarks, Gluons and Lattices* (Cambridge: Cambridge University Press)
Rothe H J 1992 *Lattice Gauge Theories. An Introduction* (Singapore: World Scientific)
Montvay I and Münster G 1994 *Quantum Fields on a Lattice* (Cambridge: Cambridge University Press)
[3] Kawamoto N 1981 *Nucl. Phys.* B **190** 617
[4] Fröhlich J and King C 1987 *Nucl. Phys.* B **290** 157
[5] Brezin E and Gross D J 1980 *Phys. Lett.* B **97** 120
Kluberg-Stern H, Morel A, Napoly O and Petersson B 1981 *Nucl. Phys.* B **190** 504
Kawamoto N and Smit J 1981 *Nucl. Phys.* B **192** 100
Aoki S 1984 *Phys. Rev.* D **30** 2653
Aoki S 1986 *Phys. Rev.* D **33** 2399
[6] Blairon J M, Brout R, Englert F and Greensite J 1981 *Nucl. Phys.* B **180** 439
Berg B and Förster D 1981 *Phys. Lett.* B **106** 323
Berg B, Billoire A and Förster D 1982 *Lett. Math. Phys.* **6** 293
Martin O 1983 *Phys. Lett.* B **130** 411
Martin O and Siu B 1983 *Phys. Lett.* B **131** 419
Gausterer H and Lang C B 1985 *Z. Phys.* C **28** 475
Zwanziger D 1986 *Nucl. Phys.* B **275** 706
[7] Gabrielli E, González-Arroyo A and Pena C 1999 $N = 1$ Supersymmetric Yang–Mills on the lattice at strong coupling *Int. J. Mod. Phys.* A to be published
(Gabrielli E, González-Arroyo A and Pena C 1999 *Preprint* hep-th/9902209)
[8] González-Arroyo A and Pena C 1999 $SU(N)$ group integration for gauge fields in the adjoint representation at large N *J. High Energy Phys.* JHEP9(1999)007
(González-Arroyo A and Pena C 1999 *Preprint* hep-th/9908026)